Handout 5: The Bombshell

Philosophy 691: Conditionals
Northern Illinois University ⋆ Fall 2011
Geoff Pynn

CONDITIONALIZATION

1. Bennett gives a proof (which he attributes to Hájek) of a key theorem (which he calls 'If-And') that relies on the notion of a probability function that "results from conditionalization." I will give a proof of that theorem that does not rely on this notion. But you should know what it means anyway.

2. We've been using the term "conditionalization on A" to name the psychological process of (hypothetically) setting your credence in A to 1 and adjusting the rest of your credences conservatively and naturally to reflect that change. Formally, "conditionalization on A" is an operation on a probability function that takes every $P(X)$ and sets it to $P(X|A)$. That this formal operation can be seen as an idealization of the relevant psychological process should be obvious. A bit more precisely:

3. If P is a probability function and $P(A) > 0$, let P_A be the function that *results from P by conditionalization on A* iff for all X, $P_A(X) = P(X|A)$. If P_A satisfies AX1, AX2, and AX3, then it is a probability function. And it does. The proofs are simple, but worth checking for yourself:

   (a) P_A satisfies AX1.

   PROOF. For all X in P's domain, $P(X \& A) \geq 0$ (AX1). So $P(X \& A)/P(A) \geq 0$, hence $P(X|A) \geq 0$, hence $P_A(X) \geq 0$.

   (b) P_A satisfies AX2.

   PROOF. If X is a logical truth, then $P(X) = 1$ (AX2). If $P(X) = 1$, then $P(X|A) = 1$ (TH6). So $P_A(X) = 1$.

   (c) P_A satisfies AX3.

   PROOF. First, note that

   $$P((X \vee Y)|A) = \frac{P((X \vee Y) \& A)}{P(A)} = \frac{P((X \& A) \vee (Y \& A))}{P(A)}$$

   Suppose that X and Y are inconsistent. Then $(X \& A)$ and $(Y \& A)$ are inconsistent. So by AX3 the above continues:

   $$= \frac{P(X \& A) + P(Y \& A)}{P(A)} = \frac{P(X \& A)}{P(A)} + \frac{P(Y \& A)}{P(A)} = P(X|A) + P(Y|A)$$

   Thus $P_A(X \vee Y) = P_A(X) + P_A(Y)$.

   Obviously, if the three axioms are satisfied by P_A, then all of the theorems we proved about probability functions from those axioms are true of P_A.

4. Here it's important to know that P_A is a probability function, because the Bennett / Hájek proof of 'If-and' applies The Equation to a function that results from conditionalization on P, and then uses what happens in that function to prove stuff about what's happening back in P. If the function resulting from conditionalization were not a probability function, it would not be legitimate to use The Equation to derive a fact about it. (The same is true of Lewis's own somewhat cryptic and very compressed instructions on how to prove it).

5. Bennett says some confusing (or just wrong) things about the relationship between the informal and formal notions of conditionalization. On page 61, he casually treats them as equivalent. On pp. 63-64, he says that Lewis's triviality proof "relies on a principle about what should happen when someone comes to be absolutely certain of something," but this is wrong. All it relies on is the formal notion of conditionalization, together with the assumption that if The Equation is true of P, then it's true of a probability function reached from conditionalization on P.

IF-AND

Lewis's triviality proof employs IF-AND: If $P(A \,\&\, B) > 0$, then $P((A \to C)|B) = P(C|(A \,\&\, B))$.

The Bennett / Hájek proof of If-And on page 62 relies on conditionalization. But we can also prove If-And by simply assuming:

IMPORT-EXPORT. $(A \,\&\, B) \to C$ is logically equivalent to $A \to (B \to C)$.

Bennett defends Import-Export directly on pp. 98-99. The road to If-And is quite straightforward:

$$\begin{aligned} P((A \to C)|B) &= P(B \to (A \to C)) && \text{The Equation} \\ &= P((B \,\&\, A) \to C) && \text{Import-Export, TH3} \\ &= P(C|(A \,\&\, B)) && \text{The Equation} \end{aligned}$$

Given what is about to come, it is surprising that Bennett wants to defend If-And / Import-Export. We'll return to this towards the end of this handout.

TRIVIALITY

First, two intuitively obvious lemmas with easy proofs:

LE3 If X entails Y, then $P(Y|X) = 1$ (provided $P(X) > 0$).

PROOF. $P(Y|X) = P(X \,\&\, Y)/P(X)$. If X entails Y, then X and $(X \,\&\, Y)$ are logically equivalent, and (by TH3) $P(X \,\&\, Y) = P(X)$. In such a case, $P(Y|X) = P(X)/P(X) = 1$.

LE4 If X entails $\neg Y$, then $P(Y|X) = 0$ (provided $P(X) > 0$).

PROOF. If X entails $\neg Y$, then Y and $(Y \,\&\, \neg X)$ are logically equivalent, and so $\dfrac{P(Y \,\&\, X)}{P(X)} = \dfrac{P(Y \,\&\, \neg X \,\&\, X)}{P(X)} = 0$.

Assume that $P(A \,\&\, C), P(A \,\&\, \neg C) > 0$, and hence that $P(A), P(C), P(\neg C) > 0$.

1. Lewis's (first) triviality result:

$$\begin{aligned} P(A \to C) &= P((A \to C) \,\&\, C) + P((A \to C) \,\&\, \neg C) && \text{TH4 (Addition Theorem)} \\ &= P((A \to C)|C)P(C) + P((A \to C)|\neg C)P(\neg C) && \text{LE1 (MVRF)} \\ &= P(C|(A \,\&\, C))P(C) + P(C|(A \,\&\, \neg C))P(\neg C) && \text{If-And} \\ &= (1)P(C) + (0)P(\neg C) && \text{LE3, LE4} \\ &= P(C) \end{aligned}$$

This is bad enough for fans of The Equation, because it implies that for a huge class of "interesting" indicatives (i.e., any in which $A \,\&\, C$ and $A \,\&\, \neg C$ both have non-zero probabilities), The Equation implies that $A \to C$ is exactly as probable as C itself. As Bennett says, this result "deprives conditionals of all their force" (p. 63). His example is good: let A = ***I become terminally and painfully ill*** and let C = ***I shall kill myself***. Then the probability of "If I become terminally and painfully ill, I shall kill myself" is the same as the probability of "I shall kill myself". Wrong answer.

But worse, The Equation tells us that if $P(C) = P(A \to C)$, then $P(C) = P(C|A)$. This is Lewis's first triviality result: that when $P(A \,\&\, C), P(A \,\&\, \neg C) > 0$, The Equation implies that $P(C) = P(C|A)$.

To see why this is absurd, we can use Lewis's example. You are about to toss a fair die. Let A = *an even number comes up* and let C = *the six comes up*. $P(A \,\&\, C)$ and $P(A \,\&\, \neg C)$ are both greater than zero. So given what we've just proved, $P(C) = P(C|A)$: the probability that the six will come up is exactly the same as the probability that the six will come up conditional on an even number's coming up. But $1/6 \neq 1/3$: wrong answer.

Why is this called a "triviality result"? Short answer: because it shows that no probability function satisfying The Equation can assign non-trivial probabilities. Longer answer: see Lewis, p. 300, or see the related result from Milne.

2. Interestingly, as Milne's paper shows, If-And and The Equation entail the equivalence of $P(A \to C)$ and $P(A \supset C)$ (provided $P(A \,\&\, C) > 0$). I repeat his proof in more expository form here. First, supposing that $P(A) > 0$:

$$\begin{array}{rcll} P(A \supset C) & = & P(\neg A \vee (A \,\&\, C)) & \text{Logical equivalence of } A \supset C \text{ and } \neg A \vee (A \,\&\, C)\text{, TH3} \\ & = & P(\neg A) + P(A \,\&\, C) & \text{AX3} \\ & = & P(\neg A) + P(C|A)P(A) & \text{LE2 (MVRF)} \\ & \geq & P(C|A)P(\neg A) + P(C|A)P(A) & \text{Because } P(C|A) \leq 1 \\ & = & P(C|A)(P(\neg A) + P(A)) & \text{Factoring} \\ & = & P(C|A) & \text{TH1} \\ & = & P(A \to C) & \text{The Equation} \end{array}$$

Thus if $P(A) > 0$, $P(A \supset C) \geq P(A \to C)$. Next, supposing that $P(A \,\&\, C) > 0$:

$$\begin{array}{rcll} P(A \to C|(A \supset C)) & = & P(C|(A \,\&\, (A \supset C))) & \text{If-And} \\ & = & P(C|(A \,\&\, C)) & \text{Logical equivalence of } A \,\&\, (A \supset C) \text{ and } A \,\&\, C\text{, TH3} \\ & = & 1 & \text{LE3} \end{array}$$

Then,

$$\begin{array}{rcll} P(A \to C) & = & P((A \to C) \,\&\, (A \supset C)) + P((A \to C) \,\&\, \neg(A \supset C)) & \text{TH4 (Addition Theorem)} \\ & \geq & P((A \to C) \,\&\, (A \supset C)) & \text{AX1} \\ & = & P((A \to C)|(A \supset C))P(A \supset C) & \text{LE2 (MVRF)} \\ & = & P(A \supset C) & P((A \to C)|(A \supset C)) = 1 \end{array}$$

So if $P(A \,\&\, C) > 0$, $P(A \to C) \geq P(A \supset C)$.

Taken together, these two results tell us that if $P(A \,\&\, C) > 0$, $P(A \to C) = P(A \supset C)$. Far from being a touchstone for developing an *alternative* to the horseshoe analysis, The Equation collapses into it.

As before, this is bad enough for fans of The Equation. But as Milne goes on to show, this result has absurd consequences. Looking at the proof that $P(A \supset C) \geq P(A \to C)$, ask yourself: what are the cases where $P(A \supset C) = P(A \to C)$? Answers:

(a) First case: when $P(C|A) = 1$. Then, by TH2, $P(A \supset C) = P(C|A) = P(A \supset C)$.

(b) Second case: when $P(\neg A) = 0$ and $P(A) = 1$. Then, by the third line of the proof, $P(A \supset C) = 0 + P(C|A) \times 1$.

(c) No other cases. If $P(C|A) < 1$ and $P(\neg A) > 0$, then the sum on the fourth line of the proof must be a smaller quantity than that on the third: the first summand on the third line is larger than the first summand on the fourth, but the second summand on each line is the same.

So given that $P(A \supset C) = P(A \to C)$ whenever $P(A \,\&\, C) > 0$, we now know that whenever $P(A \,\&\, C) > 0$, either $P(\neg A) = 0$ and thus $P(A) = 1$, or else $P(C|A) = 1$. Milne:

> "Put another way, if $0 < P(A) < 1$ and $0 < P(A \,\&\, C)$ then $P(C|A) = 1$. Since $P(C|A) = 0$ when $P(A) > 0$ and $P(A \,\&\, C) = 0$, we have our Basic Triviality Result:
>
> "the function $P(\cdot|A)$ is two-valued, i.e., takes only the values 0 and 1, when $0 < P(A) < 1$."

That is a trivial function indeed.

RESPONSES

1. Bennett's suggestions:
   (a) Restrict permissible embeddings of $\rightarrow$. You could bar things of the form $X \,\&\, (Y \rightarrow Z)$.
   (b) Follow van Fraassen in treating $\rightarrow$ as a three-place operator between A, C, and something else: his candidate for the third place, as reported by Bennett: "the speaker's epistemic state" (p. 71). Bennett says "it would be depressing for the study of conditionals generally if [van Fraassen] were right." ???

2. A response Bennett strangely doesn't suggest: deny If-And! This would require denying Import-Export. Not a small cost, but certainly not a *crazy* idea. In fact, there's a slick little argument from Gibbard (in "Two Recent Theories of Conditionals," which Bennett cites elsewhere but doesn't discuss in this connection) that Import-Export *implies the* HORSESHOE *analysis*! (My version of) Gibbard's argument makes three assumptions:
   (a) $(A \,\&\, C) \rightarrow C$ is a logical truth
   (b) Import-Export: $(A \,\&\, B) \rightarrow C$ is logically equivalent to $A \rightarrow (B \rightarrow C)$
   (c) $\rightarrow$-to-$\supset$: $A \rightarrow C$ entails $A \supset C$

   Now, the argument. Given the assumptions, (1)-(3) are logically equivalent:

(1)	$(A \,\&\, C) \rightarrow C$	A logical truth
(2)	$((A \supset C) \,\&\, A) \rightarrow C)$	(1), substitution of logical equivalents
(3)	$(A \supset C) \rightarrow (A \rightarrow C)$	(2), Import-Export

   From (3), we can derive:

(4)	$(A \supset C) \supset (A \rightarrow C)$	(3), $\rightarrow$-to-$\supset$

   Since (4) is derived from a logical truth, it is a logical truth. So $A \supset C$ entails $A \rightarrow C$. A HORSESHOE-hater looking to deny one of the assumptions should find (b) the best candidate for dismissal.

HÁJEK'S ARGUMENT

Bennett's discussion of Hájek's "Probabilities of Conditionals—Revisited" is good; you can also read Hájek's paper for yourself without much formal expertise. Here is a nice passage from a forthcoming manuscript where Hájek describes the earlier result in fairly intuitive terms ("The Fall of Adams' Thesis," http://philrsss.anu.edu.au/profile/alan-hájek):

> "Consider a fair 3-ticket lottery, and the Boolean algebra generated by the three sentences 'ticket i wins' for $i = 1, 2, 3$.[1] Let P be the natural function defined on this algebra that assigns probability $1/3$ to each of these sentences. It follows that each member of the Boolean algebra has a probability that is a multiple of $1/3$. However, various conditional probabilities are not multiples of $1/3$—for example, $P(\text{ticket 1 wins}|\text{ticket 1 wins or ticket 2 wins}) = 1/2$. So there are conditional probabilities that find no match among the unconditional probabilities. On the other hand, every unconditional probability trivially has a match among the conditional probabilities: for all X, $P(X) = P(X|T)$, where T is a tautology. So P has more distinct conditional probability values than distinct unconditional probability values.
>
> "In ["Probabilities of Conditionals—Revisited"] I showed that this result generalizes: any non-trivial finite-ranged probability function has more distinct conditional probability values than distinct unconditional probability values. This means that the function's unconditional probabilities cannot all be matched with its conditional probability values. A fortiori, this means that its unconditional probabilities *of conditionals* cannot all be matched with its conditional probability values (given that probabilities of conditionals are probabilities of their truth). There will always be some conditional probability that finds no match among the unconditional probabilities, and this will be a counterexample to [The Equation]: it will be a conditional probability of the form $P(B|A)$ that does not equal $P(A \rightarrow B)$ (or indeed anything of the form '$P(X)$').

[1] Don't be scared by the phrase 'Boolean algebra': read it as meaning 'set of sentences including those three such that if S_1 and S_2 are in the set, then so are $\neg S_1$, $\neg S_2$, $S_1 \,\&\, S_2$, and $S_1 \vee S_2$.

> "We may picture the situation poignantly as follows. Take any non-trivial probability function P with finite range. Imagine a dance, for which various men and women have entry tokens. Suppose that for each distinct value of $P(\cdot|\cdot)$, there is exactly one man with that value written on his token, and for each distinct value of $P(\cdot \to \cdot)$, there is exactly one woman with that value written on hers. There are no other men or women at the dance. It is a rule that for any couple that dances, the woman must have the same number on her token as her partner does. (I assume here that each couple consists of a woman and a man.) [The Equation] promises that everyone has a partner to dance with. The result shows that this is not so—there is at least one unmatched man who must remain a wallflower. For example, in the dance corresponding to the lottery above, the man with $1/2$ on his token will be a wallflower."

Note that Hájek's (1989) argument depends on the assumption that for every case in which $P(A) > 0$, if $P(A)$ and $P(C)$ are defined then so is $P(A \to C)$. And a fan of The Equation as Bennett states it should welcome this assumption: since whenever $P(A) > 0$, $P(C|A)$ is defined, there had better not be a case where $P(A) > 0$ but $P(A \to C)$ is undefined!

But this suggests a different way that a defender of The Equation might go in response to all of these arguments, which would be to restrict which propositions can be joined by $\to$. This would require reframing The Equation as a much more qualified claim, like:

THE RESTRICTED EQUATION. For all cases where $A \to C$ is admissible, $P(A \to C) = P(C|A)$ (provided $P(A) > 0$).

Note that disallowing conditionals with conditional consequents would *ipso facto* constitute a rejection of If-And, thus undermining the Lewis-inspired arguments. Hájek argues in the manuscript linked above that his (1989) result would still make trouble for folks who take this approach; I recommend that paper if you are interested in pursuing this issue further.